The D_n Ruijsenaars–Schneider model

2001 J. Phys. A: Math. Gen. 34 7579

(http://iopscience.iop.org/0305-4470/34/37/311)

J. Phys. A: Math. Gen. **34** (2001) 7579–7589 PII: S0305-4470(01)22311-7

The D_n Ruijsenaars–Schneider model

Kai Chen and Bo-yu Hou

Institute of Modern Physics, Northwest University, Xi'an 710069, People's Republic of China

E-mail: kai@phy.nwu.edu.cn and byhou@phy.nwu.edu.cn

Received 9 February 2001, in final form 11 July 2001
Published 7 September 2001
Online at stacks.iop.org/JPhysA/34/7579

Abstract

The Lax pair of the Ruijsenaars–Schneider model with interaction potential of trigonometric type based on D_n Lie algebra is presented. We give a general form for the Lax pair and prove partial results for small n. Liouville integrability of the corresponding system follows a series of involutive Hamiltonians generated by the characteristic polynomial of the Lax matrix. The rational case appears as a natural degeneration and the non-relativistic limit exactly leads to the well-known Calogero–Moser system associated with D_n Lie algebra.

PACS numbers: 02.20.−a, 03.65.Nk, 11.10.Lm

1. Introduction

Since a relativistic version of the Calogero–Moser (CM) model was first introduced by Ruijsenaars and Schneider [1–3], much interest has been focused on this model and its non-relativistic counterpart. It is a completely integrable many-body Hamiltonian system describing a one-dimensional n-particle system with pairwise interaction. The study has led to fascinating mathematics and applications from lattice models in statistical physics [4, 5], to the field theory and gauge theory [6], e.g. to the Seiberg–Witten theory [7], etc. For a review see [8–10], and references therein.

Recently, the Lax pairs for the CM models in various root systems have been constructed by Olshanetsky and Perelomov [11], using reduction on symmetric space, further given by Inozemtsev in [12]. Later, D'Hoker and Phong [13] succeeded in constructing the Lax pairs with spectral parameter for each of the finite-dimensional Lie algebra, as well as the introduction of untwisted and twisted Calogero–Moser systems. Bordner *et al* [14–16] give two types of universal realization for the Lax pairs associated with all of the Lie algebra: the root type and the minimal type, with and without spectral parameters. Even for all of the Coxeter group, the construction has been obtained in [17]. All of them do not apply the reduction method under which condition one will confront some obstruction [18] but by using pure Lie algebra construction. In [18], Hurtubise and Markman utilize the so called 'structure group', which combines semi-simple group and the Weyl group, to construct the CM systems associated with the Hitchin system, which to some degree generalizes the results of references [13–17]. Furthermore, the quantum version of the generalization has been developed in [19, 20] at least for degenerate potentials of trigonometry after the works of Olshanetsky and Perelomov [21].

Regarding the RS model, only the Lax pair of the A_{N-1} type RS model was obtained [2, 5, 22–25] and succeeded in recovering it by applying Hamiltonian reduction procedure on two-dimensional current group [26]. Although the commutative operators for RS model based on various type Lie algebra have been given by Komori and co-workers [27], Diejen [28, 29] and Hasegawa *et al* [4, 30], the Lax integrability (or the Lax representation) of the other type RS model is still an open problem [7].

In a recent work [31, 32], we have succeeded in constructing the Lax pairs for the rational and trigonometric C_n and BC_n RS systems. Following that, the r-matrix structure for them have been derived by Avan *et al* in [33]. Moreover, we give the more general elliptic C_n and BC_n RS systems in [34] and calculate their spectral curves. In this paper, we concentrate on generalizing the construction to the D_n-type trigonometric Ruijsenaars–Schneider model. It turns out that there surely exists a Lax pair for this system. By revealing the symmetry property of this model, we shall give a general form of the Lax pair for generic n and verify its rationality at least for small n such as $n = 2, 3, 4, 5, 6$. Its integrability in the Liouville sense is also depicted by giving n involutive integrals of motion. We also perform its non-relativistic limit that coincides exactly with the previous known result for the D_n Calogero–Moser system. The rational degeneration of this system is also remarked.

The paper is organized as follows. The basic materials of the D_n RS model are introduced in section 2, where we propose a self-consistent dynamical system associating with the root system of D_n. This includes construction of Hamiltonian for the D_n RS system together with its symmetry analysis, etc. The main results are shown in section 3. In the section, we present a Lax pair and obtain an explicit general form for the Lax pair by imposing some additional symmetry constraints. Section 4 is devoted to deriving the non-relativistic counterpart, the Calogero–Moser model. Following are some remarks for the degenerate limit of rational case. We conclude with some remarks on our constructions in the last section.

2. Model and equations of motion

Let us first review the basic materials about the D_n RS model. Although much progress has been made for generalization of the RS model [3, 27, 28, 31, 32, 34], there is no result for the system which associates with the root system of D_n. Even up to now we do not know how to define its Hamiltonian. But now we will give a reasonable definition for this system which will be seen later.

In terms of the canonical variables p_i, x_i $(i, j = 1, \ldots, n)$ enjoying in the canonical Poisson bracket

$$\{p_i, p_j\} = \{x_i, x_j\} = 0 \qquad \{x_i, p_j\} = \delta_{ij} \tag{2.1}$$

we give firstly the Hamiltonian of D_n RS system

$$H = \sum_{i=1}^{n} \left(\mathrm{e}^{p_i} \prod_{k\neq i}^{n} (f(x_{ik}) f(x_i + x_k)) + \mathrm{e}^{-p_i} \prod_{k\neq i}^{n} (g(x_{ik}) g(x_i + x_k)) \right) \tag{2.2}$$

where

$$\begin{aligned} f(x) &:= \frac{\sin(x-\gamma)}{\sin(x)} \\ g(x) &:= f(x)|_{\gamma\to-\gamma} \qquad x_{ik} := x_i - x_k \end{aligned} \tag{2.3}$$

and γ denotes the coupling constant. Then the canonical equations of motion could be

$$\dot{x}_i = \{x_i, H\} = \mathrm{e}^{p_i} b_i - \mathrm{e}^{-p_i} b_i' \tag{2.4}$$

$$\begin{aligned}\dot{p}_i = \{p_i, H\} = &\sum_{j\neq i}^{n} \left(\mathrm{e}^{p_j} b_j (h(x_{ji}) - h(x_j + x_i)) + \mathrm{e}^{-p_j} b_j' (\hat{h}(x_{ji}) - \hat{h}(x_j + x_i))\right) \\ &- \mathrm{e}^{p_i} b_i \left(\sum_{j\neq i}^{n} (h(x_{ij}) + h(x_i + x_j))\right) \\ &- \mathrm{e}^{-p_i} b_i' \left(\sum_{j\neq i}^{n} (\hat{h}(x_{ij}) + \hat{h}(x_i + x_j))\right)\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &h(x) := \frac{\mathrm{d}\ln f(x)}{\mathrm{d}x} \qquad \hat{h}(x) := \frac{\mathrm{d}\ln g(x)}{\mathrm{d}x} \\ &b_i = \prod_{k\neq i}^{n} (f(x_i - x_k) f(x_i + x_k)) \\ &b_i' = \prod_{k\neq i}^{n} (g(x_i - x_k) g(x_i + x_k)) .\end{aligned} \tag{2.6}$$

Here, of course $x_i = x_i(t)$, $p_i = p_i(t)$ and the superimposed dot denotes t-differentiation.

For the convenience of analysis of symmetry, let us first give vector representation of D_n Lie algebra. Introducing an n-dimensional orthonormal basis of $\mathbb{R}^n$,

$$e_j \cdot e_k = \delta_{j,k} \qquad j, k = 1, 2, \ldots, n \tag{2.7}$$

then the sets of roots Δ and vector weights Λ of D_n are

$$\Delta = \{\pm(e_j - e_k), \pm(e_j + e_k) \colon\ j, k = 1, 2, \ldots, n \text{ and } j < k\} \tag{2.8}$$

$$\Lambda = \{e_j, -e_j \colon j = 1, 2, \ldots, n\}. \tag{2.9}$$

The dynamical variables are canonical coordinates $\{x_j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets of equation (2.1). In a general sense, we denote them by n-dimensional vectors x and p,

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \qquad p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n.$$

So that the scalar products of x and p with the roots $\alpha \cdot x$, $p \cdot \beta$, etc can be defined. The Hamiltonian of equation (2.2) can be rewritten as

$$H = \frac{1}{2} \sum_{\mu\in\Lambda} \left(\exp(\mu \cdot p) \prod_{\Delta\ni\beta=\mu-\nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta\ni\beta=-\mu+\nu} g(\beta \cdot x) \right). \tag{2.10}$$

Here, the condition $\Delta \ni \beta = \mu - \nu$ means that the summation is over roots β such that for $\exists \nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta. \tag{2.11}$$

So does for $\Delta \ni \beta = -\mu + \nu$.

From the above-mentioned data, we can see that the definition for the Hamiltonian is reasonable and well-defined whose form equation (2.2) or equation (2.10) is similar to the one given in [31, 32, 34].

3. Construction of the Lax pair

In this section, we concentrate our treatment to the explicit form of the Lax pair for the D_n RS system. Therefore, some previous results, as well as new results, could now be obtained in a more straightforward manner by using the Lax pair.

3.1. Derivation of the Lax matrix for the D_n RS model

Similar to the definitions of the Lax matrixes for the C_n and BC_n RS models given in [32], we suppose the Lax matrix for the D_n RS model is one $2n \times 2n$ matrix as follows:

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.1}$$

where A, B, C, D are $n \times n$ matrixes (hereafter, we use the indices $i, j = 1, \ldots, n$)

$$\begin{aligned} A_{ij} &= \mathrm{e}^{p_j} b_j \frac{\sin\gamma}{\sin(x_{ij}+\gamma)} \qquad D_{ij} = \mathrm{e}^{-p_j} b'_j \frac{\sin\gamma}{\sin(x_{ji}+\gamma)} \\ B_{ij} &= (1-\delta_{ij})\mathrm{e}^{-p_j} b'_j \frac{\sin\gamma}{\sin(x_i+x_j+\gamma)} + \delta_{ij}\mathrm{e}^{-p_i}\frac{b'_i}{w_i}\tilde{B}_{ii} \\ C_{ij} &= (1-\delta_{ij})\mathrm{e}^{p_j} b_j \frac{\sin\gamma}{\sin(-x_i-x_j+\gamma)} + \delta_{ij}\mathrm{e}^{p_i}\frac{b_i}{w'_i}\tilde{C}_{ii}. \end{aligned} \tag{3.2}$$

Here, the notations of w_i, w'_i, v_i are

$$\begin{aligned} w_i &:= \prod_{j\neq i}^{n} \sin(x_i+x_j+\gamma)\sin(x_{ij}+\gamma) \\ w'_i &:= \prod_{j\neq i}^{n} \sin(x_i+x_j-\gamma)\sin(x_{ij}-\gamma) \\ v_i &:= \prod_{j\neq i}^{n} \sin(x_i+x_j)\sin(x_{ij}) \end{aligned} \tag{3.3}$$

and $\tilde{B}_{ii}, \tilde{C}_{ii}$, the diagonal part of block matrixes B and C, are unknown and have to be solved later.

In order to obtain the explicit form of $\tilde{B}_{ii}, \tilde{C}_{ii}$, we also assume the inverse of L the following $2n \times 2n$ matrix (similar to the form for the C_n and BC_n cases)

$$L^{-1} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \tag{3.4}$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are $n \times n$ matrixes

$$\begin{aligned} \hat{A}_{ij} &= \mathrm{e}^{-p_i} b'_j \frac{-\sin\gamma}{\sin(x_{ij}-\gamma)} \qquad \hat{D}_{ij} = \mathrm{e}^{p_i} b_j \frac{-\sin\gamma}{\sin(x_{ji}-\gamma)} \\ \hat{B}_{ij} &= (1-\delta_{ij})\mathrm{e}^{-p_i} b_j \frac{-\sin\gamma}{\sin(x_i+x_j-\gamma)} + \delta_{ij}\mathrm{e}^{-p_i}\frac{b_i}{w'_i}\tilde{C}_{ii} \\ \hat{C}_{ij} &= (1-\delta_{ij})\mathrm{e}^{p_i} b'_j \frac{-\sin\gamma}{\sin(-x_i-x_j-\gamma)} + \delta_{ij}\mathrm{e}^{p_i}\frac{b'_i}{w_i}\tilde{B}_{ii}. \end{aligned} \tag{3.5}$$

If we impose an additional condition for $\tilde{B}_{ii}$ and $\tilde{C}_{ii}$ as

$$\tilde{C}_{ii} = \tilde{B}_{ii}|_{\gamma\to-\gamma} \tag{3.6}$$

then the equation

$$L \cdot L^{-1} = Id \tag{3.7}$$

can be solved and the solution reads

$$\begin{aligned}\tilde{B}_{ii} &= \frac{w_i}{b_i'}\left(1 - b_i b_i' - \sin^2\gamma \sum_{j\neq i}^{n}\left(\frac{b_i' b_k}{\sin^2(x_{ik}+\gamma)} + \frac{b_i' b_k'}{\sin^2(x_i + x_k + \gamma)}\right)\right)^{1/2}\\ \tilde{C}_{ii} &= \tilde{B}_{ii}|_{\gamma\to-\gamma}.\end{aligned}\tag{3.8}$$

So that

$$\begin{aligned}B_{ii} &= \mathrm{e}^{-p_i}\frac{b_i'}{w_i}\tilde{B}_{ii}\\ C_{ii} &= \mathrm{e}^{p_i}\frac{b_i}{w_i'}\tilde{C}_{ii} = B_{ii}|_{\gamma\to-\gamma,\, p_i\to -p_i}.\end{aligned}\tag{3.9}$$

Remarks. The above solution of equations (3.8) and (3.9) is obtained only by the diagonal part of equation (3.7). It is not easy to verify if the off-diagonal part is consistent with the diagonal part due to complicated functional relations. But for small n such as $n = 2, 3, 4, 5, 6$ we can surely check that it is the very unique solution. In addition, it is unfortunate that we are not able to give more simple forms for B_{ii} and C_{ii}. Here only for $n = 2, 3, 4$ we work out the following results to shed a light on its appearance:

- for $n = 2$

$$\begin{aligned}\tilde{B}_{ii} &= \sin^2\gamma\\ \tilde{C}_{ii} &= \sin^2\gamma\end{aligned}\tag{3.10}$$

- for $n = 3$

$$\begin{aligned}\tilde{B}_{ii} &= \frac{1}{2}\sin^2\gamma\left(w_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j+\gamma)\,\sin(x_{ij}+\gamma)} + v_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j)\,\sin(x_{ij})}\right)\\ \tilde{C}_{ii} &= \frac{1}{2}\sin^2\gamma\left(w_i'\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j-\gamma)\,\sin(x_{ij}-\gamma)}\; v_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j)\,\sin(x_{ij})}\right)\\ &= \tilde{B}_{ii}|_{\gamma\mapsto-\gamma}\end{aligned}\tag{3.11}$$

- for $n = 4$

$$\begin{aligned}\tilde{B}_{ii} &= \frac{1}{2}\sin^2\gamma\left(w_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j+\gamma)\sin(x_{ij}+\gamma)}\right.\\ &\qquad\left.+\,v_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j)\sin(x_{ij})} - \sin^2\gamma\sin^2(2x_i+\gamma)\right)\\ \tilde{C}_{ii} &= \frac{1}{2}\sin^2\gamma\left(w_i'\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j-\gamma)\sin(x_{ij}-\gamma)}\right.\\ &\qquad\left.+\,v_i\sum_{j\neq i}^{n}\frac{1}{\sin(x_i+x_j)\,\sin(x_{ij})} - \sin^2\gamma\sin^2(2x_i-\gamma)\right) = \tilde{B}_{ii}|_{\gamma\mapsto-\gamma}.\end{aligned}\tag{3.12}$$

With the Lax matrix L of equation (3.1), we could rewrite the Hamiltonian as

$$H = \sum_{j=1}^{n} (e^{p_j} b_j + e^{-p_j} b'_j) = \operatorname{tr} L. \tag{3.13}$$

The involutive n Hamiltonians can be generated by the characteristic polynomial of the Lax matrix

$$\det(L - v \cdot Id) = \sum_{j=0}^{2n} (-1)^j (v^j + v^{2n-j}) H_j + (-v)^n H_n \tag{3.14}$$

with

$$\{H_i, H_j\} = 0 \qquad i, j = 1, 2, \ldots, n. \tag{3.15}$$

For example, for $n = 2$,

$$\det(L - v \cdot Id) = v^4 - H_1 v^3 + H_2 v^2 - H_1 v + 1 \tag{3.16}$$

the function-independent Hamiltonian flows H and H_2 are

$$\begin{aligned} H_1 = H = {} & e^{p_1} f(x_{12}) f(x_1 + x_2) + e^{-p_1} g(x_{12}) g(x_1 + x_2) \\ & + e^{p_2} f(x_{21}) f(x_2 + x_1) + e^{-p_2} g(x_{21}) g(x_2 + x_1) \end{aligned} \tag{3.17}$$

$$\begin{aligned} H_2 = {} & 2\Big(f(x_{12}) g(x_{12}) + f(x_1 + x_2) g(x_1 + x_2) \Big) \\ & + e^{p_1 + p_2} f(x_1 + x_2)^2 + e^{-p_1 - p_2} g(x_1 + x_2)^2 \\ & + e^{p_1 - p_2} f(x_{12})^2 + e^{p_2 - p_1} g(x_{12})^2 + \text{const} \end{aligned} \tag{3.18}$$

where const $= -2$. For $n = 3$, we have

$$\det(L - v \cdot Id) = v^6 - H_1 v^5 + H_2 v^4 - H_3 v^3 + H_2 v^2 - H_1 v^1 + 1 \tag{3.19}$$

and

$$H_1 = H = \sum_{i=1}^{3} \left(e^{p_i} \prod_{k \neq i}^{3} f(x_{ik}) f(x_i + x_k) + e^{-p_i} \prod_{k \neq i}^{3} g(x_{ik}) g(x_i + x_k) \right)$$

$$H_2 = \tilde{H}_2 - 1 \tag{3.20}$$

$$H_3 = \tilde{H}_3 - 2H_1. \tag{3.21}$$

Here $\tilde{H}_2$ and $\tilde{H}_3$ are the involutive Hamiltonians defined for the D_3 RS model by Diejen in [28]:

$$\begin{aligned} H_+ = {} & e^{(-p_1 - p_2 + p_3)/2} f(-x_1 - x_2) f(-x_1 + x_3) f(-x_2 + x_3) \\ & + e^{(-p_1 + p_2 - p_3)/2} f(-x_1 + x_2) f(-x_1 - x_3) f(x_2 - x_3) \\ & + e^{(p_1 + p_2 + p_3)/2} f(x_1 + x_2) f(x_1 + x_3) f(x_2 + x_3) \\ & + e^{(p_1 - p_2 - p_3)/2} f(x_{12}) f(x_{13}) f(-x_2 - x_3) \end{aligned} \tag{3.22}$$

$$\begin{aligned} H_- = {} & e^{(-p_1 - p_2 - p_3)/2} f(-x_1 - x_2) f(-x_1 - x_3) f(-x_2 - x_3) \\ & + e^{(-p_1 + p_2 + p_3)/2} f(-x_1 + x_2) f(-x_1 + x_3) f(x_2 + x_3) \\ & + e^{(p_1 - p_2 + p_3)/2} f(x_{12}) f(x_1 + x_3) f(-x_2 + x_3) \\ & + e^{(p_1 + p_2 - p_3)/2} f(x_1 + x_2) f(x_{13}) f(x_{23}) \end{aligned} \tag{3.23}$$

$$\tilde{H}_2 = H_+ H_- \tag{3.24}$$

$$\tilde{H}_3 = H_+^2 + H_-^2. \tag{3.25}$$

We verify that these H_i and $\tilde{H}_j$ strictly Poisson commute each other, which ensures the complete integrability of the D_2 and D_3 RS models (in the Liouville sense).

3.2. M operator associating with L

By comparing the symmetry of the D_n RS model and BC_n one, we propose the following ansatz for M operator associating with the Lax matrix L so that they satisfy

$$\dot{L} = \{L, H\} = [M, L]. \tag{3.26}$$

Suppose M to be another $2n \times 2n$ matrix with the form

$$M = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \tag{3.27}$$

where entries of M are

$$\begin{aligned}
\mathcal{A}_{ij} &= \cot(x_{ij}) \left(\mathrm{e}^{p_j} b_j \frac{\sin\gamma}{\sin(x_{ij}+\gamma)} + \mathrm{e}^{-p_i} b'_j \frac{\sin\gamma}{\sin(x_{ij}-\gamma)} \right) \qquad j \neq i \\
\mathcal{D}_{ij} &= \cot(x_{ji}) \left(\mathrm{e}^{-p_j} b'_j \frac{\sin\gamma}{\sin(x_{ji}+\gamma)} + \mathrm{e}^{p_i} b_j \frac{\sin\gamma}{\sin(x_{ji}-\gamma)} \right) \qquad j \neq i \\
\mathcal{B}_{ij} &= \cot(x_i + x_j) \left(\mathrm{e}^{-p_j} b'_j \frac{\sin\gamma}{\sin(x_i+x_j+\gamma)} + \mathrm{e}^{-p_i} b_j \frac{\sin\gamma}{\sin(x_i+x_j-\gamma)} \right) \qquad j \neq i \\
\mathcal{C}_{ij} &= \cot(-x_i - x_j) \left(\mathrm{e}^{p_j} b_j \frac{\sin\gamma}{\sin(-x_i-x_j+\gamma)} + \mathrm{e}^{p_i} b'_j \frac{\sin\gamma}{\sin(-x_i-x_j-\gamma)} \right) \qquad j \neq i \\
\mathcal{A}_{ii} &= -\left(\sum_{j\neq i}^{n} \frac{\mathcal{A}_{ij}}{\cos(x_{ij})} + \sum_{j=1}^{n} \frac{\mathcal{B}_{ij}}{\cos(x_i+x_j)} \right) \\
\mathcal{D}_{ii} &= -\left(\sum_{j\neq i}^{n} \frac{\mathcal{D}_{ij}}{\cos(x_{ji})} + \sum_{j=1}^{n} \frac{\mathcal{C}_{ij}}{\cos(-x_i-x_j)} \right).
\end{aligned} \tag{3.28}$$

If we impose $\mathcal{B}_{ii}$, $\mathcal{C}_{ii}$ an additional symmetry condition with

$$\mathcal{B}_{ii} = \mathrm{e}^{2p_i} \mathcal{C}_{ii} \tag{3.29}$$

verbose but straightforward calculations of equations

$$\dot{L}_{ii} = \{L_{ii}, H\} = ([M, L])_{ii} = \sum_{k\neq i}^{2n} (M_{ik} L_{ki} - L_{ik} M_{ki}) \tag{3.30}$$

would yield

$$\begin{aligned}
\mathcal{B}_{ii} = {} & \frac{\sin^2\gamma}{C_{ii}\mathrm{e}^{-p_i} - B_{ii}\mathrm{e}^{p_i}} \mathrm{e}^{-p_i} \left(-2b_i b'_i \sum_{j\neq i}^{n} \left(\frac{\cos(x_i+x_j)}{\sin(x_i+x_j+\gamma)\sin(x_i+x_j-\gamma)} \right.\right. \\
& \left. + \frac{\cos(x_i - x_j)}{\sin(x_i-x_j+\gamma)\sin(x_i-x_j-\gamma)} \right) \\
& + \sum_{j\neq i}^{n} \left(\frac{b_i b'_j \cot(x_{ik})}{\sin^2(x_{ik}-\gamma)} + \frac{b'_i b_j \cot(x_{ik})}{\sin^2(x_{ik}+\gamma)} + \frac{b_i b_j \cot(x_i+x_j)}{\sin^2(x_i+x_j-\gamma)} \right. \\
& \left.\left. + \frac{b'_i b'_j \cot(x_i+x_j)}{\sin^2(x_i+x_j+\gamma)} \right) \right) \\
\mathcal{C}_{ii} = {} & \mathrm{e}^{2p_i} \mathcal{B}_{ii}.
\end{aligned} \tag{3.31}$$

As for the explicit expression of $\mathcal{B}_{ii}, \mathcal{C}_{ii}$, we have more simple forms for small n:

- for $n = 2$,

$$\begin{aligned} \mathcal{B}_{ii} &= 0 \\ \mathcal{C}_{ii} &= 0 \end{aligned} \tag{3.32}$$

- for $n = 3$,

$$\begin{aligned} \mathcal{B}_{ii} &= \frac{2}{v_i} \mathrm{e}^{-p_i} \cos\gamma \cos(2x_i) \sin^3\gamma \\ \mathcal{C}_{ii} &= \frac{2}{v_i} \mathrm{e}^{p_i} \cos\gamma \cos(2x_i) \sin^3\gamma \\ &= \mathrm{e}^{2p_i} \mathcal{B}_{ii} \end{aligned} \tag{3.33}$$

- for $n = 4$,

$$\begin{aligned} \mathcal{B}_{ii} &= \frac{2}{v_i} \mathrm{e}^{-p_i} \cos\gamma \cos(2x_i) \sin^3\gamma \left(2\cos x_i \sin^2\gamma + \sum_{j\neq i}^{n} \sin(x_i + x_j) \sin(x_{ij}) \right) \\ \mathcal{C}_{ii} &= \frac{2}{v_i} \mathrm{e}^{p_i} \cos\gamma \cos(2x_i) \sin^3\gamma \left(2\cos x_i \sin^2\gamma + \sum_{j\neq i}^{n} \sin(x_i + x_j) \sin(x_{ij}) \right) \\ &= \mathrm{e}^{2p_i} \mathcal{B}_{ii}. \end{aligned} \tag{3.34}$$

We have checked that L, M satisfy the Lax equation of equation (3.26) which is equivalent to the equations of motion equations (2.4) and (2.5) at least for $n = 2, 3, 4, 5, 6$ with the help of the computer.

4. Non-relativistic limit to the Calogero–Moser system

It is natural that we must verify if the non-relativistic limit is correct. The procedure can be achieved by rescaling $p_i \longmapsto \beta p_i, \gamma \longmapsto \beta\gamma$ while letting $\beta \longmapsto 0^+$ (here, 0^+ is to avoid undefinable limit of $\mathcal{B}_{ii}$ and $\mathcal{C}_{ii}$ when $n = 2$), and making a canonical transformation

$$p_i \longmapsto p_i + \gamma \left(\sum_{k\neq i}^{n} (\cot(x_{ik}) + \cot(x_i + x_k)) \right) \tag{4.1}$$

such that

$$L \longmapsto Id + \beta L_{\mathrm{CM}} + O(\beta^2) \tag{4.2}$$

$$M \longmapsto 2\beta M_{\mathrm{CM}} + O(\beta^2) \tag{4.3}$$

and

$$H \longmapsto 2n + 2\beta^2 H_{\mathrm{CM}} + O(\beta^2). \tag{4.4}$$

L_{CM} can be expressed as

$$L_{\mathrm{CM}} = \begin{pmatrix} A_{\mathrm{CM}} & B_{\mathrm{CM}} \\ -B_{\mathrm{CM}} & -A_{\mathrm{CM}} \end{pmatrix} \tag{4.5}$$

where

$$\begin{aligned} (A_{\mathrm{CM}})_{ij} &= \delta_{ij} p_i + (1 - \delta_{ij}) \frac{\gamma}{\sin(x_{ij})} \\ (B_{\mathrm{CM}})_{ij} &= (1 - \delta_{ij}) \frac{\gamma}{\sin(x_i + x_j)}. \end{aligned} \tag{4.6}$$

M_{CM} is

$$M_{\mathrm{CM}} = \begin{pmatrix} \mathcal{A}_{\mathrm{CM}} & \mathcal{B}_{\mathrm{CM}} \\ \mathcal{B}_{\mathrm{CM}} & \mathcal{A}_{\mathrm{CM}} \end{pmatrix} \tag{4.7}$$

where

$$\begin{aligned} (\mathcal{A}_{\mathrm{CM}})_{ij} &= -\delta_{ij} \sum_{k\neq i}^{n} \left(\frac{\gamma}{\sin^2 x_{ik}} + \frac{\gamma}{\sin^2(x_i + x_k)} \right) + (1-\delta_{ij}) \frac{\gamma \cos(x_{ij})}{\sin^2 x_{ij}} \\ (\mathcal{B}_{\mathrm{CM}})_{ij} &= (1-\delta_{ij}) \frac{\gamma \cos(x_i + x_j)}{\sin^2(x_i + x_j)} \end{aligned} \tag{4.8}$$

which coincides with the form given in [11, 14] with the difference of a constant diagonalized matrix.

The Hamiltonian of the D_n-type CM model can be given by

$$H_{\mathrm{CM}} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 - \gamma^2 \sum_{k<i}^{n} \left(\frac{1}{\sin^2 x_{ik}} + \frac{1}{\sin^2(x_i + x_k)} \right) = \frac{1}{4} \operatorname{tr} L^2. \tag{4.9}$$

The L_{CM}, M_{CM} satisfy the Lax equation

$$\dot{L}_{\mathrm{CM}} = \{L_{\mathrm{CM}}, H_{\mathrm{CM}}\} = [M_{\mathrm{CM}}, L_{\mathrm{CM}}]. \tag{4.10}$$

Remarks. As far as the forms of the Lax pair for the rational-type RS and CM systems are concerned, we can get them by making the following substitutions

$$\begin{aligned} \sin x &\to x \\ \cos x &\to 1 \end{aligned} \tag{4.11}$$

for all of the above statements.

5. Summary and discussions

In this paper, we have presented the Lax pair for the classical n-particle trigonometric D_n Ruijsenaars–Schneider model together with its rational limit. We give one explicit form of the Lax pair for small n such as 2, 3, 4 and show the involutive Hamiltonians could be generated by the corresponding Lax matrix. For generic n we have constructed the Lax pair and given a general form to it though lacking in a complete proof. But its correctness could be checked at least for $2 \leqslant n \leqslant 6$. In the non-relativistic limit, this system naturally leads to the well-known Calogero–Moser system associated with the root system of D_n.

Actually, our original aim is to expand our constructions to the dynamical systems associated with all of the root systems. As suggested in [35] and [26], A_{n-1} RS model appeared in the Hamiltonian reduction procedure applied to the cotangent bundle over centrally extended current group while the cotangent bundle over the centrally extended current algebra was used to obtain the elliptic Calogero–Moser model [36, 37]. It is natural to expect similar results for other root systems. Unfortunately, we fail in the corresponding constructions for the systems associated with the root systems other than A_{n-1}. In fact, as was analysed in [18], there are several obstructions to extend the constructions. Alternatively, they used the so called ‘structure group’, which related to Weyl reflections, to process symplectic reduction to construct the CM systems associated with the Hitchin system where the embedding was not even a group but a semi-direct product of groups. Moreover, one has the BC_n CM and RS systems but they do not even correspond to groups. So the more general and elegant method to universal construction for the RS systems must combine all of characters appeared in previous results and get over the obstructions mentioned above.

On the other hand, a more concrete method is to use pure algebraic construction, which has made great success for CM systems [13, 14–17]. In the present paper, we try to follow this idea and work out partial result for D_n RS system where some formulas such as equations (3.6) and (3.9) have revealed some characters of Weyl reflections. Though we have not obtained universal description of this system, we hope these results would reveal some essential ingredient for its integrability and shed some light on universal characters for generic RS systems. At the same time, we address an interesting aspect that the reduction procedure of using 'structure group' corresponding to RS systems and fixing certain momentum map suggested in [38, 18] may be a potential method to accomplish the complete generalization for RS systems associated with all of simple Lie algebra and even to all of root systems. Moreover, the issue for getting the r-matrix structure for this model is deserved due to the success of calculation for the trigonometric BC_n RS system by Avan *et al* in [33].

Acknowledgment

One of the authors, K Chen, is grateful to professors K J Shi, L Zhao and W L Yang for their encouragement and valuable discussions.

References

[1] Ruijsenaars S N M and Schneider H 1986 *Ann. Phys.* **170** 370
[2] Ruijsenaars S N M 1987 *Commun. Math. Phys.* **110** 191
[3] Ruijsenaars S N M 1988 *Commun. Math. Phys.* **115** 127
[4] Hasegawa K 1997 *Commun. Math. Phys.* **187** 289
[5] Nijhoff F W, Kuznetsov V B, Sklyanin E K and Ragnisco O 1996 *J. Phys. A: Math. Gen.* **29** L333
[6] Nekrasov N 1998 *Nucl. Phys.* B **531** 323
[7] Braden H W, Marshakov A, Mironov A and Morozov A 1999 *Nucl. Phys.* B **558** 371
[8] D'Hoker E and Phong D H 1999 Lectures on supersymmetric Yang–Mills theory and integrable systems *Preprint* hep-th/9912271
[9] Marshakov A 1998 *Seiberg–Witten Theory and Integrable Systems* (Singapore: World Scientific)
[10] Gorsky A and Mironov A 2000 integrable many-body systems and gauge theories *Preprint* hep-th/0011197
[11] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 314
Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Boston, MA: Birkhäuser)
[12] Inozemtsev V I 1989 *Lett. Math. Phys.* **17** 11
[13] D'Hoker E and Phong D H 1998 *Nucl. Phys.* B **530** 537
[14] Bordner A J, Corrigan E and Sasaki R 1998 *Prog. Theor. Phys.* **100** 1107
[15] Bordner A J, Sasaki R and Takasaki K 1999 *Prog. Theor. Phys.* **101** 487
[16] Bordner A J and Sasaki R 1999 *Prog. Theor. Phys.* **101** 799
[17] Bordner A J, Corrigan E and Sasaki R 1999 *Prog. Theor. Phys.* **102** 499
[18] Hurtubise J C and Markman E 1999 Calogero–Moser systems and Hitchin systems *Preprint* math.AG/9912161
[19] Bordner A J, Manton N S and Sasaki R 2000 *Prog. Theor. Phys.* **103** 463
[20] Khastgir S P, Pocklington A J and Sasaki R 2000 *J. Phys. A: Math. Gen.* A **33** 9033
[21] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
[22] Bruschi M and Calogero F 1987 *Commun. Math. Phys.* **109** 481
[23] Krichever I and Zabrodin A 1995 *Usp. Math. Nauk* **50** 3
[24] Suris Y B 1996 Why are the rational and hyperbolic Ruijsenaars–Schneider hierarchies governed by the same R-operators as the Calogero–Moser ones? *Preprint* hep-th/9602160
[25] Suris Y B 1997 *Phys. Lett.* A **225** 253
[26] Arutyunov G E, Frolov S A and Medvedev P B 1997 *J. Math. Phys.* **38** 5682
[27] Komori Y and Hikami K 1998 *J. Math. Phys.* **39** 6175
[28] van Diejen J F 1994 *J. Math. Phys.* **35** 2983
[29] van Diejen J F 1995 *Compositio. Math.* **95** 183

[30] Hasegawa K, Ikeda T and Kikuchi T *J. Math. Phys.* **40** 4549

[31] Chen K, Hou B Y and Yang W L 2001 *Chin. Phys.* **10** 550

[32] Chen K, Hou B Y and Yang W L 2000 *J. Math. Phys.* **41** 8132

[33] Avan J and Rollet G 2000 BC_n Ruijsenaars–Schneider models: R-matrix structure and Hamiltonians *Preprint* hep-th/0008174

[34] Chen K, Hou B Y and Yang W L 2000 The Lax pair for elliptic C_n and BC_n Ruijsenaars–Schneider models and their spectral curves *Preprint* hep-th/0011145 (2001 *J. Math. Phys.* **42** at press)

[35] Gorsky A and Nekrasov N 1995 *Nucl. Phys.* B **436** 582

[36] Gorsky A and Nekrasov N 1994 Elliptic Calogero–Moser system from two-dimensional current algebra *Preprint* hep-th/9401021

[37] Arutyunov G E and Medvedev P B 1995 Generating equation for r-matrices related to dynamical systems of Calogero type *Preprint* hep-th/9511070

[38] Nekrasov N 1996 *Commun. Math. Phys.* **180** 587